arithmetic based upon two units instead of one—*i.e.*, the arithmetic of couples or complex quantities—could be completely and consistently represented by choosing as axes whereon the separate units were counted, the two perpendicular axes of Cartesian geometry. An attempt to extend this geometrical representation into space led Hamilton to the invention of his method, Gauss having very early satisfied himself that within the limits of ordinary algebra no further extension was necessary or possible.

The examination into fundamental principles was not limited in the mind of Gauss to those of algebra: he early applied himself likewise to those of geometry and of dynamics. The great French mathematicians, such as Legendre and Lagrange, were also occupied with such speculations. They have been carried on all through the century, but have only towards the end of the period been brought into connection and shown to be of importance for the general progress of mathematics. The secluded, and for a long time unappreciated, labours of isolated but highly original thinkers have accordingly

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considered as merely special instances. This has now been abundantly proved through the writings of mathematicians in all countries, among whom I will only mention Hankel and Dr V. Schlegel in Germany, Clifford, Prof. Henrici, and latterly Mr Whitehead in England, Prof. Peano in Italy, and M. Burali Forti in France. See on the whole subject, on the fate of Grassmann and of his great work, V. Schlegel, 'Die Grassmann'sche Ausdehnungslehre,' Leipzig, 1896; also, by the same author, a short biography of Grassmann (Leipzig, Brockhaus, 1878). A complete edition of Grassmann's works is being published by Teubner. Those who are interested in seeing how the notions underlying the directional calculus are gradually becoming clarified, and the terminology and notation settled, may read with profit the controversy carried on in the pages of 'Nature,' vols. xlvii. and xlviii., between Prof. Macfarlane, Willard Gibbs, Mr O. Heaviside, Mr A. M'Aulay, and Dr Knott; also Dr Larmor's review of Hayward's 'Algebra of Coplanar Vectors' (vol. xlvii. p. 266), and Sir R. S. Ball's reference to the 'Ausdehnungslehre' of Grassmann (vol. xlviii. p. 391, 1893).