Before Weierstrass, Cauchy and Riemann had attempted to define the vague term "function" or mathematical dependence. Both clung to the graphical representation so common and so helpful in analysis since Descartes invented it. We have, of course, in abstract science, a right to begin with any definition we choose. Only the definition must be such that it

remarkable tract on "Oscillating functions," in which he drew attention to the existence of functions which admit of an integral, but where the existence of a differential coefficient remains doubtful. In fact, it appears that the question as to the latter had never been raised; the only attempt in this direction being that of Ampère in 1806, which failed (Hankel, p. 7). Hankel in his original investigation showed that a continuous curve might be supposed to be generated by the motion of a point which oscillated to and fro, these oscillations at the limit becoming infinitely numerous and infinitely small: a curve thus generated would present what he called "a condensation of singularities" at every point, but would possess no definite direction, hence also no differential coefficient. The arguments and illustrations of Hankel have been criticised and found fault with. He nevertheless deserves the credit of having among the first attempted "to gain a firm footing on a slippery road which had only been rarely trodden" (p. 8). In this tract (which is reprinted in 'Math. Ann.,' vol. xx.), as well as in his valuable article on "Limit" (Ersch und Gruber, 'Encyk.,' vol. xc. p. 185, art. "Grenze"), Hankel did much to establish clearly the essential point on which depends the entire modern revolution in our ideas regarding the foundations of the so-called infinitesimal calculus; reverting to the idea of a "limit," both in the definition of the derived function (limit of a ratio) and of the integral (limit of a sum) as contained in the writings both of Newton and Leibniz, but obscured by the method of "Fluxions" of the former and the method of "Infinitesimals" of the latter. Lagrange and Cauchy had begun this revolution, but it was not consistently and generally carried through till the researches of Riemann, Hankel, Weierstrass, and others made rigorous definitions necessary and generally accepted. It is, however, well to note that in this country A. de Morgan very early expressed clear views on this subject. Prof. Voss, in his excellent chapter on the Differential and Integral Calculus ('Encyk. Math. Wiss.,' vol. ii. i. p. 54, &c.), calls the later period the period of the purely arithmetical examination of infinitesimal conceptions, and says (p. 60), "The purely arithmetical definition of the infinitesimal operations which is characteristic of the present critical period of mathematics has shown that most of the theorems established by older researches, which aimed at a formal extension of method, only possess a validity limited by very definite assumptions." Such assumptions were tacitly made by earlier writers, but not explicitly stated.