the labours of the two great analysts is nowhere better shown than in the special manner in which Weierstrass succeeded in strengthening the foundations¹ on which much of Riemann's work rests.

The labours of the great analysts—Gauss, Cauchy, Riemann, and Weierstrass—all tended to increase our

publication Weierstrass withdrew from the press an extensive memoir which he had presented in the year 1857 to the Berlin Academy, because, as he himself says (Weierstrass, 'Math. Werke,' vol. iv. p. 10): "Riemann published a memoir on the same problem which rested on entirely different foundations from mine, and did not immediately reveal that in its results it agreed completely with my own. The proof of this required investigations which were not quite easy, and took much time; after this difficulty had been removed a radical remodelling of my dissertation seemed necessary," &c. &c. The mutual influence of Riemann's and Weierstrass's work is also referred to by Weierstrass in a letter to Prof. Schwarz, dated 1875, in which he utters what he calls his confession of faith: "The more I ponder over the principles of the theory of functions-and I do this incessantly - the stronger grows my conviction that it must be built up on the foundation of algebraical truths, and that, therefore, to employ for the proof of simple and fundamental algebraical theorems the 'transcendental,' if I may say so, is not the correct way, however enticing prima vista the considerations may be by which Riemann has discovered many of the most important properties of algebraical functions. It is a matter of course that every road must be open to the searcher as long as he seeks; it is only a question of the systematic demonstration" (Weierstrass, 'Werke,' vol. ii. p. 235).

¹ This refers mainly to Weierstrass's investigation of the principle called by Riemann "Dirichlet's principle," but which had been stated already with great generality by Thomson (Lord Kelvin) in the The validity of this year 1847. method depended on a certain minimum theorem. Weierstrass has shown that the existence of such a minimum is not evident, and that the argument used is not conclusive. He laid before the Berlin Academy, in the year 1870, a communication giving a test-case to prove that Dirichlet's method was not generally valid ('Werke,' vol. ii. p. 49). "Through this," Prof. Klein says (loc. cit., p. 67), "a great part of Riemann's developments become invalidated. Nevertheless the far - reaching results which Riemann bases upon the principle are all correct, as was shown later on exhaustively and with all rigour by Carl Neumann and H. A. Schwarz. Indeed we must come to the conclusion that Riemann himself arrived at these theorems by a physical intuition, and only afterwards resorted to the principle referred to in order to have a consistent mathematical line of reasoning" (loc. cit., p. 67). See on this also Poincaré (loc. cit., pp. 10 and 15), who gives other instances where the work of Weierstrass supported that of Riemann.

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