Bolyai in Hungary, Lobatchevski in Kasan, Grassmann in Stettin. Most of these were unknown to each other. However, near the beginning of the last third of the century three distinct publications created a great stir in the mathematical world, brought many scattered but cognate lines of reasoning together, and made them mutually fertile and suggestive. These three were—*first*, the publication in 1860 of Gauss's correspondence with Schumacher, in which two letters of the former, dated May and July 1831,[1] became known, where he referred to his extensive but unwritten and unfinished speculations on the foundations of geometry and the theorem which refers to the sum of the angles in a triangle. The *second* was the publication in 1867 of the first and only part of Hermann Hankel's "Lectures on the Complex Numbers and their Functions."[2] The *third* was the posthumous publication in the same year of Riemann's paper, dated 1854,[3] "On the Hypotheses which lie at the Foundation of Geometry." Almost simultaneously there appeared the first of Helmholtz's two important papers[4] on the

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[1] See 'Briefwechsel zwischen Gauss und Schumacher,' ed. Peters, 1860, vol. ii. pp. 260, 268.

[2] The small volume contains so much original and historical matter that I have on several occasions referred to it. See above, pp. 645, 653.

[3] Riemann, 'Math. Werke,' 1st ed., p. 254 *sqq.*

[4] The first publication of Helmholtz was a lecture on "the actual foundations of geometry," which he delivered on the 22nd May 1868 to the Medical Society at Heidelberg. This communication, which

referred to investigations carried on for many years,—notably in connection with the theory of the colour-manifold, — was occasioned by the publication of Riemann's paper in the 'Transactions' of the Göttingen Society. He had heard of this through Schering, to whom he wrote on the 21st April 1868 before having seen Riemann's paper: "I have myself been occupied with the same subject during the last two years, in connection with my researches in physiological optics. . . . I now see, from the few hints which you give as to the