an introduction to the advanced theories of Gauss and Riemann; and for this purpose he went back to the unnoticed labours of Grassmann in Germany, to the writings of Peacock and De Morgan in England, and incidentally introduced into Germany the elaborate algebra of quaternions, invented and practised by Hamilton twenty years before that time. The papers of Riemann and Helmholtz similarly showed the necessity of a thorough investigation of the principles and foundations of ordinary or Euclidean geometry, and showed how consistent systems of geometry could be elaborated on other than Euclidean axioms. Only
53. Non-Euclidean geometry.
from that moment, in fact, did it become generally recognised that already, a generation before, two independent treatises on elementary geometry had been published in which the axiom of parallel lines was dispensed with and consistent geometrical systems developed. These were contained—as already stated—in the 'Kasan Messenger,' under date 1829 and

ing only a short paper by Dr F. Gregory on Symbolical Algebra in the Edinburgh 'Transactions.' Whilst Hankel was delivering lectures on these fundamentals, Weierstrass in Berlin was likewise in the habit of introducing his lectures on the Theory of Analytic Functions by a discussion of the theory of Complex Numbers. This introduction was published, with Weierstrass's permission, in the year 1872 by Dr E. Kossak (in a programme of the Friedrichs-Werder Gymnasium), after lectures delivered by Weierstrass in 1865-66. To what extent Hankel may have been influenced by Weierstrass's lectures, which he seems to have attended after leaving Göttingen, is uncertain, for in spite of his very extensive references he does not mention Weierstrass. In Kossak's 'Elemente der Arithmetik' the term "permanence of formal rules" is not used, but the treatment of the extended arithmetic is carried on along the same lines—*i.e.*, not by an attempt to represent the complex quantities, but on the ground of maintaining the rules which govern the arithmetic of ordinary numbers. Great importance is also attached to the principle of inversion as having shown itself of value in the theory of elliptic functions, and being not less valuable in arithmetic. As stated above (p. 640, note), this principle is also insisted on by Peacock.