1830, the author being Lobatchevski; and in the appendix to an Introduction to Geometry, published by Wolfgang Bolyai at Maros Vasarheli, a town of Transylvania, the appendix being by the author's son, Johann Bolyai. The elder Bolyai having been a friend and correspondent of Gauss, and his speculations evidently of the same nature as those indicated by the latter in the above-mentioned correspondence, conjectures have been made as to which of the two originated the whole train of thought.[1] The independent investigations of Riemann and Helmholtz started from a differ-

[1] See above, p. 652, note. What is important from our point of view in the investigations of both Riemann and Helmholtz lies in the following points: First, Neither Riemann nor Helmholtz refers to the non-Euclidean geometry of Lobatchevski or Bolyai. This is not surprising in the case of Helmholtz, whose interest was originally not purely mathematical; in fact, we may incidentally remark how, in spite of his profound mathematical ability, he on various occasions came into close contact with mathematical researches of great originality and importance without recognising them—*e.g.*, the researches of Grassmann and Plücker. As regards Riemann, his paper was read before Gauss, who certainly knew all about Bolyai, and latterly also about Lobatchevski, of whom he thought so highly that he proposed him as a foreign member of the Göttingen Society. Gauss could therefore easily have pointed out to Riemann the relations of his speculations with his own and those of the other mathematicians named. Since the publication of the latest volume of Gauss's works, it has become evident that Gauss corresponded a good deal, and more than one would have supposed from reading Sartorius's obituary memoir, on the subject of non-Euclidean (astral or imaginary) geometry, notably with Gerling; and that several contemporary mathematicians, such as Schweikart, came very near to Gauss's own position. Second, although Riemann, and subsequently also Helmholtz, made use of the term "manifold" (*Mannigfaltigkeit*), it does not appear in the course of their discussion that they considered the space-manifold from any other than a metrical point of view. In fact, the manifold becomes in their treatment a magnitude (*Grösse*). It is true that Riemann does refer to certain geometrical relations not connected with magnitude but only with position, as being of great importance. These two points through which the researches of Riemann and Helmholtz stand in relation to other, and at the time isolated, researches, were dwelt on, the first by Beltrami, and the second by Cayley and Prof. Klein.