There exists, moreover, an analogy between the manner in which these novel and extended ideas have been historically introduced and the mode of reasoning which led Sir W. R. Hamilton to the invention of a new and extended algebra—the algebra of quaternions. This analogy becomes evident if we study the small volume of Hermann Hankel, which appeared about the same time as Riemann's and Beltrami's fundamental geometrical dissertations.

55. Generalised conceptions.

The extension of Hamilton was only possible by dropping one of the fundamental principles of general arithmetic, the commutative principle of multiplication, which is symbolically expressed by saying that $a \times b$ is equal to $b \times a$. By assuming that $a \times b$ is equal to $-b \times a$, Hamilton founded a new general arithmetic on an apparently paradoxical principle. Similarly Lobatchevski and Bolyai constructed new geometries by dropping the axiom of parallel lines. Hankel made clear the significance of the new algebra, Riemann and Beltrami that of the new geometry. The practical performance anticipated and led up to the theoretical or philosophical exposition of the underlying principles. But there was a third instance in which a new science had been created by abandoning the conventional way of looking at things. This was the formation of a consistent body of geometrical teaching by disregarding the metrical properties and studying only the positional or projective properties, following Monge and Poncelet. The two great minds who worked out this geometry independently of the conception of number or measurement, giving a purely geometrical definition of distance and number, were Cayley in Eng-