land and Von Staudt in Germany. It was reserved for Prof. Felix Klein of Göttingen to show how the generalised notions of distance introduced into geometry by Cayley and Von Staudt opened out an understanding of the three geometries of Euclid, of Lobatchevski, and of Riemann.[1] We have to go back to the purely projective properties of space to understand these different possibilities. Lobatchevski attacked the problem practically, Riemann analytically, Klein geometrically. Through the labours of Klein the subject has arrived at a certain finality. And what was still wanting after he had written his celebrated memoir (which was approved and

56. Klein's exposition.

[1] See the note on p. 714, above; also 'Math. Ann.,' vol. iv. p. 573, and vol. vi. p. 112. Prof. Klein—following a usage in mathematical language — distinguishes three different geometries, the hyperbolic, the elliptic, and the parabolic geometry, corresponding to the possession by the straight line at infinity of two real or two imaginary (that is, none) or two coincident points. The whole matter turns upon the fact that, although metrical relations of figures are in general changed by projection, there is one metrical relation—known in geometry as the "anharmonic ratio" (in German *Doppelverhältniss*)—which in all projective transformations remains unchanged. As this anharmonic ratio of points or lines can be geometrically constructed without reference to measurement (Von Staudt, 'Geometrie der Lage,' 1847 and 1857), a method is thus found by which, starting from a purely descriptive property or relation, distance and angles—*i.e.*, metrical quantities—can be defined. Some doubts have been expressed whether, starting from the purely projective properties of space and building up geometry in this way (arriving at the metrical properties by the construction suggested by Von Staudt), the ordinary idea of distance and number is not tacitly introduced from the beginning. This may be of philosophical, but is not of mathematical, importance, as the main object in the mathematical treatment is to gain a starting-point from which the several possible consistent systems of geometry can be deduced and taken into view together. See on this point, *inter alia*, Cayley's remarks in the appendix to vol. ii. of 'Collected Works' (p. 604 *sqq.*), also Sir R. S. Ball's paper (quoted there), and more recently the discussion on the subject in Mr Bertrand Russell's 'Essay on the Foundations of Geometry' (1897. p. 31, &c.; p. 117, &c.) See also the same author's article on non-Euclidean Geometry in the supplement of the 'Ency. Brit.,' vol. xxviii.