commented on by Cayley) was later on supplied in consequence of a suggestion of his. The researches of Riemann, and still more those of Helmholtz, had not merely a mathematical, they had also a logical and a psychological, meaning. Space was conceived to be a threefold - extended manifold. There are other manifolds besides space—such, for instance, as the threefold-extended manifold of colours. Helmholtz came from the study of this manifold to that of space. Now the question arises as to the conditions or data which are necessary and sufficient for the foundations of a science like geometry. We have seen that the axiom of parallel lines is not required; we have also seen that the notion of distance and number can be generalised. What other data remain which cannot be dispensed with? Helmholtz had attempted to answer this question. But neither he nor Riemann had considered the possibility of a purely projective geometry. Now it is the merit of Prof. Klein to have seen that there exists a purely algebraical method by which this problem can be attacked. This is the method of groups referred to above, and 57. Sophus Lie.
applied by Sophus Lie to assemblages of continuously variable quantities. Klein was one of the first to recognise the power of this new instrument. He saw that the space problem was a problem of transformations, the possible motions in space forming a group with definite elements (the different freedoms of motion) which were continuously variable—*i.e.*, in infinitesimal quantities—and which returned into themselves under certain well-defined conditions. They possessed, moreover, in the maintenance of distance the algebraic property of in-