variance. He also expressed some doubt regarding the logical consistency of the assumptions of Helmholtz. Sophus Lie undertook this investigation, and thus brought the logical side of the labours of Riemann and Helmholtz to a final conclusion.[1] This is one of the celebrated instances where the rigorous algebraical methods have detected flaws in the more intuitional or purely geometrical process, and extended our knowledge of hidden possibilities.

But there is yet another branch of the great science of number, form, and interdependence, the principles and foundations of which had been handed down from earlier ages, where the critical and sifting process of the nineteenth century has led to an expansion and revolution of our fundamental ideas. Here also, as in so many other directions, the movement begins with Gauss. Hitherto I have spoken mainly of algebra or general arithmetic, of geometry, of the connections of both in the

[1] "Lie was early made aware by Klein and his "program" that the space problem belonged to the theory of groups. . . . Ever since 1880 he had been pondering over these questions; he published his views first in 1886 on the occasion of the Berlin meeting of natural philosophers. Helmholtz's conception was itself unconsciously (but remarkably so, inasmuch as it dates from 1868) one belonging to the theory of groups, trying, as it did, to characterise the groups of the sixfold infinite motions in space, which led to the three geometries, in comparison with all other groups. He did this by fixing on the free mobility of rigid bodies—*i.e.*, on the existence of an invariant between two points as the only essential invariant. When Lie took up this problem in principle, as one belonging to the theory of groups, he recognised that for our space that part of the axiom of monodromy was unnecessary which added periodicity to the free mobility round a fixed axis. . . . The value of these investigations lies mainly in this, that they permit of our fixing for every kind of geometry the most appropriate system of axioms. . . . And they justly received in the year 1897 the first Lobatchevski prize awarded by the Society of Kasan" (M. Nöther, 'Math. Ann.,' vol. liii. p. 38). A lucid exposition of Lie's work will be found in Mr B. Russell's 'Essay,' &c., p. 47 *sqq*.