60. Generalised conception of number.

same time a metaphysical question presented itself—viz., Can such an extension into more than two dimensions be consistently and profitably carried out? Gauss had satisfied himself that it could not;[1] but the proof of this was only given in more recent times by Weierstrass, who definitely founded the whole discussion of the subject on the logical principle "that the legitimacy of introducing a number into arithmetic depends solely on the definition of such number." And this leads me to another extension in the region of number suggested by Gauss's treatment, which has also become fundamental, and, in the hands of Dirichlet, Kummer, Liouville, Dedekind, and others, has remodelled the entire science of higher arithmetic. It is based on the logical process of the

[1] A concise history of this subject is given by Kossak in the Program referred to above, p. 712, note. Gauss had promised to answer the question, "Why the relations between things which have a manifoldness of more than two dimensions would not admit of other" (than the ordinary complex numbers introduced by him) "fundamental quantities being introduced into general arithmetic?" He never redeemed his promise. In consequence of this, several eminent mathematicians, notably Hankel, Weierstrass, and Prof. Dedekind, have attempted to reply to this question, and to establish the correctness of the implied thesis according to which any system of higher complex numbers becomes superfluous and useless. Prof. Stolz, in the first chapter of the second volume of his 'Allgemeine Arithmetik,' gives an account of these several views, which do not exactly coincide. In general, however, the proof given by Weierstrass, and first published by Kossak, has been adopted. This proof is based upon the condition that the product of several factors cannot disappear except one of its factors is equal to zero. "We must, therefore, exclude from general arithmetic complex numbers consisting of three fundamental elements. This is, however, not necessary if the use of them be limited" by some special conditions (Kossak, *loc. cit.*, p. 27). In the course of the further development of this matter Weierstrass arrives at the fundamental thesis "that the domain of the elementary operations in arithmetic is exhausted by addition and multiplication, including the inverse operations of subtraction and division." "There are," says Weierstrass, "no other fundamental operations—at least it is certain that no example is known in analysis where, if an analytical connection exists at all, this cannot be analysed into and reduced to those elementary operations" (p. 29).